Hausdorff dimension of the limit sets of classical Schottky groups(Analysis of Discrete Groups)

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0. Introduction

1.1. In this paper we will consider the following problem : Given any number t satisfying $0 < t < 1$, does there exist a finitely generated Kleinian group G with the limit set $\Lambda(G)$ having infinite t - dimensional Hausdorff measure ?

In 1971, Beardon[1] gave an affirmative answer by using Hecke groups for this problem. The method of Beadon depends on a close, direct analysis of the action of the group G. Furthermore, in 1985 Phillips and Sarnak [5] showed by using the bottom of the spectrum for the Laplacian Δ (the smallest eigenvalue of Δ) that there is a Hecke group having the desired property. Here we will consider the problem by studying Fuchsian Schottky groups (Sato [7]).

We will state the method of Beardon in §1 and the method of Phillips-Sarnak in §2. In §3 we will state some results on Fuchsian Schottky groups.

1. The method of Beardon

In this section we will state the proof of the following theorem due to Beardon

THEOREM A (Beardon [1]). *Given any number t satisfying $t < 1$, there exists a finitely generated Fuchsian group G of the second kind with ∞ an ordinary point of G and with the limit set $\Lambda(G)$ having infinite t-dimensional Hausdorff measure.*

DEFINITION 1.1. Let E be any set and t a positive number. Define

$$m_{t,\delta}(E) = \inf \sum_i |I_i|^t,$$

where the infimum is taken over all coverings of E by sequences $\{I_i\}$ of sets I_i with diameter $|I_i|$ less than δ. Furthermore, we define

$$m_t(E) = \sup\{m_{t,\delta}(E) | \delta > 0\}$$

and we call $m_t(E)$ the *t-dimensional Hausdorff measure* of E.

Set $d(E) = \inf\{t | m_t(E) = 0\}$. We call $d_t(E)$ the *Hausdorff dimension* of E.

DEFINITION 1.2. A set E is said to be a *spherical Cantor set* if and only if it can be expressed in the form

$$E = \cap_{n=1}^{\infty} \cup_{i_1,\ldots,i_n=1}^{K} \Delta(i_1, \ldots, i_n)$$

where $K \geq 2$ is an integer and where the $\Delta_{i_1,\ldots,i_n}$ are closed spheres of radius $r(i_1, \ldots, i_n)$ satisfying

(1) $\Delta(i_1, \ldots, i_n) \supset \Delta(i_1, \ldots, i_n, i_{n+1})$,

(2) $\Delta(1), \cdots, \Delta(K)$ are mutually disjoint,

(3) there exists a constamt A $(0 < A < 1)$ such that

$$r(i_1, \ldots, i_n, i_{n+1}) \geq A r(i_1, \ldots, i_n) \qquad (i_{n+1} = 1, 2, \ldots, K),$$

(4) there exists a constant B $(0 < B < 1)$ such that

$$\rho(\Delta(i_1, \ldots, i_n, j), \Delta(i_1, \ldots, i_n, k)) \geq B r(i_1, \ldots, i_n)$$

$$(j,k=1,2,\ldots,K, j\neq k)$$

where

$$\rho(S,T)=\inf\{|s-t||s\in S, t\in T\}$$

DEFINITION 1.3. Let $P(z)=z+2(1+\varepsilon)$ and $E(z)=-1/z$. We call the group $G[\varepsilon]$ generated by $P(z)$ and $E(z)$ a *Hecke group*.

1.2 Since the point ∞ is a limit point of $G[\varepsilon]$, we conjugate $G[\varepsilon]$ by $A\in$ Möb such that ∞ is an ordinary point of $AG[\varepsilon]A^{-1}$. We denote by $\Lambda(G)$ the limit set of a group G. For simplicity, we denote by Λ_ε the limit set of a Hecke group $G[\varepsilon]$.

LEMMA 1.1.

$$d(A(\Lambda_\varepsilon))\geq d(A(\Lambda_\varepsilon\cap[-1,1]))\geq d(\Lambda_\varepsilon\cap[-1,1]).$$

REDUCTION 1. It suffices to show that for sufficiently small ε, $d(\Lambda_\varepsilon\cap[-1,1])\geq t$.

NOTATION.

$Q:=\{z||z|\leq 1\}$,

$V_n(z):=EP^n(z)\quad(n\neq 0)$,

$V(n_1,n_2,\ldots,n_k)(z):=V_{n_1}V_{n_2}\cdots V_{n_k}(z)\quad(n_j\neq 0)$,

$Q(n_1,n_2,\ldots,n_k):=V(n_1,n_2,\ldots,n_k)(Q)$

$L_1:=\cap_{k=1}^{\infty}\cup_{V\in G_k}V(Q)$,

where $G_k=\{V(n_1,n_2,\ldots,n_k)|n_j=1,2,\ldots,K\}$.

LEMMA 1.2. *L_1 is a subset of* $\Lambda_\varepsilon \cap [-1, 1]$.

1.3. Let ϵ be a positive number and N an integer satisfying $N \geq 2$. Let Γ_1 be the set consisting of the following elements (1) and (2):

(1) (A) $V_2, V_{-2}, \ldots, V_N, V_{-N}$

(2) $V(n_1, n_2, \ldots, n_k, m)$ with

(B) $1 \leq k \leq N, n_1 = \cdots = n_k = 1$ and $2 \leq |m| \leq N$,

(B') $1 \leq k \leq N, n_1 = \cdots = n_k = -1$ and $2 \leq |m| \leq N$,

(C) $1 \leq k \leq N, n_1 = \cdots = n_k = 1$ and $m = -1$,

(C') $1 \leq k \leq N, n_1 = \cdots = n_k = -1$ and $m = 1$.

We set

$$\Gamma_{n+1} := \{UV | U \in \Gamma_n, V \in \Gamma_1\}$$

and

$$L_2 := \cap_{k=1}^{\infty} \cup_{V \in \Gamma_n} V(Q).$$

Then we have $L_2 \subset \Lambda_\epsilon \cap [-1, 1]$. Hence

$$d(L_2) \leq d(L_1) \leq d(\Lambda_\epsilon \cap [-1, 1]) \leq 1.$$

REDUCTION 2. It suffices to show that

$$\lim_{\epsilon \to \infty} \limsup_{N \to \infty} d(L_2) = 1$$

1.4. Set $\Gamma := \cup_n \Gamma_n$. We denote by $|\Delta|$ the diameter of a disc Δ.

LEMMA 1.3. *Let $J = [-1,1]$, let I be any sub-interval of J and let $U \in \Gamma$.* Then

(1) $\frac{1}{5}|I| \leq \frac{|U(I)|}{|U(J)|} \leq \frac{5}{4}|I|$

(2) *If $V \in \Gamma_1$, then* $|UV(J)| \leq \frac{5}{6}|U(J)|$.

LEMMA 1.4. *The set L_2 is a spherical Cantor set constructed from the discs $\{U(Q)|U \in \Gamma_n, n \geq 1\}$.*

LEMMA 1.5. *If θ satisfies $0 \leq \theta \leq 1$ and if*

$$\sum_{V \in \Gamma_1} |UV(Q)|^\theta \geq |U(Q)|^\theta$$

for all $U \in \Gamma$, then $d(L_2) \geq \theta$.

LEMMA 1.6. *Let $k > 1$ be any integer and let the positive numbers $\delta_1, \ldots, \delta_k, \delta$ and s satisfy $0 \leq \delta_j \leq \delta < 1$ and $0 \leq s \leq \delta_1 + \cdots + \delta_k < 1$. Then*

$$\delta_1^\theta + \cdots + \delta_k^\theta \geq 1,$$

where $\theta = 1 - (1-s)(1-\delta)^{-1}$.

1.5. We set $F := (-1,1) - \cup_{V \in \Gamma_1} V(J)$. Then

$$|U(J)| = m_1(U(F)) + \sum_{V \in \Gamma_1} |U(V(J))|.$$

By Lemma 3 we have

$$m_1((U)F)) \leq \frac{5}{4} m_1(F)|U(J)|$$

and

$$\begin{aligned} \delta_1 + \cdots + \delta_k \;&:=\; \sum_{V \in \Gamma_1} \frac{|V(J)|}{|U(J)|} = 1 - \frac{m_1(U(F))}{|U(J)|} \\ &\geq\; 1 - \frac{5}{4} m_1(F). \end{aligned}$$

We take s in Lemma 1.6 $s = 1 - \frac{5}{4}m_1(F)$, and we take $\delta = \frac{6}{5}$ by Lemma 1.3. Then

$$\theta = 1 - \frac{1-s}{1-\delta} = 1 - \frac{15}{2}m_1(F).$$

By Lemma 1.6 we have $\sum \delta_j^\theta \geq 1$, that is,

$$\sum_{V \in \Gamma_1} \frac{|UV(J)|^\theta}{|U(J)|^\theta} \geq 1.$$

Hence

$$\sum |UV(J)|^\theta \geq |U(J)|^\theta.$$

By Lemma 1.5 $d(L_2) \geq \theta$ and so

$$d(L_2) \geq 1 - \frac{15}{2}m_1(F) > 1 - 8m_1(F).$$

REDUCTION 3. It suffices to show that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} m_1(F) = 0.$$

1.6. PROOF of Theorem A. We set $T := (-1.1) - \cup_{|n|=1}^N V(n)(J)$. For convenience, define $u_n = 1$ and $v_n = -1$ for each positive integer n. Then we have

$$\begin{aligned} F - T &= \cup_{n=-1,1} V(n)(J) - \cup_{V \in \Gamma_1} V(J) \\ &= [F \cap V(1)(J)] \cup [F \cap V(-1)(J)] \end{aligned}$$

Hence

$$m_1(F) = m_1(T) + m_1[F \cap V(1)(J)] + m_1[F \cap V(-1)(J)].$$

After calculations, we have

$$m_1[F \cap V(1)(J)]$$
$$= \sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] + m_1[V(u_1, \ldots, u_{N+1})(J)]$$

Noting that both T and J are symmetrical with the imaginary axix, we have

$$m_1(F)$$
$$= m_1(T) + 2\sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] + 2m_1[V(u_1, \ldots, u_{N+1})(J)].$$

We estimate three terms on the right hand side.

(1) The first term: If we set $\mu = 2 + 2\varepsilon$, then

$$m_1(T) = \frac{2}{N\mu + 1} + 2\sum_{r=0}^{N}[\frac{1}{\mu r + 1} - \frac{1}{\mu(r+1) - 1}] < \frac{1}{N} + 6\varepsilon.$$

(2) The second term:

$$\sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] \leq \frac{3m_1(T)}{\sqrt{\varepsilon}} < \frac{3}{\sqrt{\varepsilon}}(\frac{1}{N} + 6\varepsilon).$$

(3) The last term :

$$m_1[V(u_1, \ldots, u_{N+1})(J)] \leq (\frac{p-q}{p-1})^2 \frac{m_1(J)}{p^{2(N+1)}},$$

where $p = (\mu + \sqrt{\mu^2 - 4})/2$ and $q = (\mu - \sqrt{\mu^2 - 4})/2$. Hence we have

$$m_1(F) \leq (\frac{1}{N} + 6\epsilon)(1 + \frac{6}{\sqrt{\epsilon}}) + 4(\frac{p-q}{p-1})^2 \frac{1}{p^{2(N+1)}}.$$

Therefore

$$\limsup_{N \to \infty} m_1(F) \leq 6\epsilon + 36\sqrt{\epsilon}$$

and so

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} m_1(F) = 0,$$

which is the desired result.

2. The method of Phillips-Sarnak

2.1. In this section we will state the proof of the following theorem due to Phillips-Sarnak. Let $G[\varepsilon]$ be the Hecke group defined in §1. We denote by G_μ the Hecke group $G[\varepsilon]$ with $\mu = 2 + 2\epsilon$.

THEOREM B (Phillips-Sarnak [5]). *Let $\lambda_0(G_\mu)$ be the smallest eigenvalue of the Laplacian Δ for a Hecke group G_μ. As μ ranges from 2 to ∞, $\lambda_0(G_\mu)$ increases continuously and strictly monotonically from* 0 *to* 1/4.

COROLLARY. *Let $d(G_\mu)$ be the Hausdorff dimension of the limit set of a Hecke group G_μ. As μ ranges from* 2 *to $\infty, d(G_\mu)$ decreases continuously and strictly monotonically from* 1 *to* 1/2.

This corolarry follows from Theorem B and Pattrson-Sullivan's theorem below.

2.2. Let $H = \{(x, y)|x \in R, y > 0\}$ be the upper half plane with the line element $ds^2 = (dx^2 + dy^2)/y^2$. We denote by Δ, ∇ and dV the Laplacian, gradient and volume element, respectively, with respect to the hyperbolic metric. Let Ω be an open connected subset of H. We denote by $W^1(\Omega)$ the space of functions

$$W^1(\Omega) = \{f \in L^2(\Omega)|\nabla f \in L^2(\Omega)\}.$$

The quadratic forms H and D on $W^1(\Omega)$ are defined as

$$H(f, g) := \int_{|} \Omega f\bar{g}dV$$

and

$$D(f,g) := \int_{\Omega} < \nabla f, \bar{\nabla} g > dV.$$

Here we are interested in the selfadjoint Laplasian Δ defined on $L^2(\Omega)$ with Neumann boundary condition. This means that the domain of this operator consists of the set of all functions $u \in W^1(\Omega)$ with square integrable satisfying the condition $H(\Delta u, v) = D(u, v)$ (which is equivalent to $\partial u/\partial n = 0$, where $\partial/\partial n$ is the unit outer normal derivative). We denote by $\lambda_0(\Omega)$ the bottom of the spectrum for Δ on $L^2(\Omega)$. $\lambda_0(\Omega)$ can be described variationally as

$$\lambda_0(\Omega) = \inf\{D(u) | u \in W^1(\Omega), H(u) = 1\}.$$

DEFINITION 2.1. We call a domain Ω *free* if $\lambda_0(\Omega) = 1/4$.

We remark that Ω is free if and only of the spectrum for Δ on $L^2(\Omega)$ have no discrete spectrum.

Let G be a discrete group acting on the upper half plane H. We set

$$\delta(G) := \inf\{s | \sum_{\gamma \in G} \exp(-s(\rho(z, \gamma w)) < +\infty\},$$

where $\rho(z, \gamma w)$ is the hyperbolic distance from z to γw. We call $\delta(G)$ the *exponent of convergence* of G.

Patterson-Sullivan's theorem (Patterson [4], Sullivan [8]).

(1) $\delta(G) \geq 1/2$ then $\lambda_0(G) = \delta(G)(1 - \delta(G))$.

(2) If G is geometrically finite, then $\delta(G) = d(\Lambda(G))$.

COROLLARY. *G is a geometrically finite group with $\lambda_0(G) = 0$, then $d(\Lambda(G)) = 1$.*

DEFINITION 2.3. If a domain Ω is bounded by nonoverlapping circles, then we call Ω a *Schottky domain.* We call a discrete group G a

Schottky group in the sense of Phillips-Sarnak or a *P-S Schottky group* if it has a fundamental domain which is a Schottky domain.

REMARK. A Hecke group G_μ is both a Fuchsian group of the second kind (resp.a Fuchsian group of the first kind) and a symmetric P-S Schottky group if $\mu = 2 + 2\epsilon > 2$ (resp. $\mu = 2$), where a domain Ω is symmetric if Ω is symmetric with respect to the imaginary axix.

2.5. Let G be a discrete group. We denote by $\lambda_0(G) < \lambda_1(G) \leq \cdots$ the discrete eigenvalue of Δ on the Hilbert space of G automorphic functions. We note that $\lambda_j(\Omega) \leq \lambda_j(G)$ if Ω is a fundamental domain for G.

LEMMA 2.1. *If G is a symmetric P-S Schottky group, then $\lambda_j(G) = \lambda_j(\Omega^+)$, where Ω^+ is the part of the right side of $\Omega \cap H$ with respect to the imaginary axis.*

COROLLARY. *If G_μ is a Hecke group, then $\lambda_0(G_\mu) = \lambda_0(F_\mu^+)$, where F_μ^+ is the part of the right side of the symmetric fundamental domain F_μ for G_μ with respect to the imaginary axis.*

2.6. LEMMA 2.2. *Suppose $\Omega' \to \Omega$ in H.*

(1) *If no cusp is broken in going from Ω to Ω' , then*

$$\lim \lambda_j(\Omega') = \lambda_j(\Omega).$$

(2) *If a cusp is broken and if $\Omega' \supset \Omega$ and $\Omega' \backslash \Omega$ is free, then* $\lim \lambda_0(\Omega') = \lambda_0(\Omega)$.

COROLLARY. *If G_μ is a Hecke group, then $\lambda_0(G_\mu)$ is continuous in $\mu (2 \leq \mu < \infty)$.*

LEMMA 2.3. *Let G_μ is a Hecke group. Then $\lambda_0(G_\mu) = 0$ for $\mu = 2$, that is, $d(\Lambda(G_\mu)) = 1$ for $\mu = 2$.*

LEMMA 2.4. *Suppose Ω_0 and Ω_1 are two domains with $\bar{\Omega}_1 \subset \Omega_0$ and set $\Omega_2 = \Omega_0 \backslash \bar{\Omega}_1$.*

(1) *If Ω_2 is free, then $\lambda_j(\Omega_0) \geq \lambda_j(\Omega_1)$ for all j.*

(2) *Furthermore, if Ω_0 has the finite geometric property and Ω_1 is not*

free, then $\lambda_0(\Omega_0) > \lambda_0(\Omega_1)$.

COROLLARY. *If* G_μ *is a Hecke group, then* $\lambda_0(G_\mu)$ *increase strictly monotonically in* μ $(2 \leq \mu < \infty)$

LEMMA 2.5. *For Schottky domains, if* $\Omega' \to \Omega$, *then* $\lambda_0(\Omega_k) \to \lambda_0(\Omega)$.

COROLLARY. *If* G_μ *is a Hecke group, then* $\lambda(G_\mu) \to \lambda_0(G_\infty)$ as $\mu \to \infty$.

LEMMA 2.6. *If* Ω *is a domain in* H *with at most* $[(n+4)/2]$ *sides, then* Ω *is free.*

COROLLARY. *If* G_μ *is a Hecke group, then* $\lambda_0(G_\infty) = 1/4$.

Theorem B follows from the above lemmas and corollaries.

3. Some results

3.1. In this section we will consider the problem stated in the introduction by using Fuchsian Schottky groups. Let $G =< A_1, A_2 >$ be a Schottky group generated by Möbius transformations A_1 and A_2. We define t_j $(0 < |t_j| < 1)$ in such a way that $1/t_j$ is the multiplier of A_j $(j = 1, 2)$. Let p_j and q_j be the repelling and the attracting fixed points of A_j $(j = 1, 2)$. We define $\rho \in \mathbf{C} - \{0, 1\}$ by setting $(0, \infty, 1, \rho) = (p_1, q_1, p_2, q_2)$, where (z_1, z_2, z_3, z_4) is the cross ratio of z_1, z_2, z_3 and z_4. We say $< A_1, A_2 >$ represents (t_1, t_2, ρ), or (t_1, t_2, ρ) corresponds to $< A_1, A_2 >$. There are eight kinds of classical Schottky groups of real type of genus two (see Sato [6] for detail). Here we will consider the Hausdorff dimension of the limit sets of two kinds of classical Schottky groups, that is, Fuchsian Schottky groups.

DEFINITION 3.1. Let (t_1, t_2, ρ) be the point corresponding to a Schottky group $G =< A_1, A_2 >$.

(1) G is the first kind if $t_1 > 0, t_2 > 0$ and $\rho > 0$.

(2) G is the fourth kind if $t_1 > 0, t_2 > 0$ and $\rho < 0$.

We call the above Schottky group (1) or (2) a *Fuchsian Schottky group* of genus two.

We denote by $R_I \mathfrak{S}_2^0$ ($resp. R_{IV} \mathfrak{S}_2^0$) the space of all classical Schottky groups of type I (resp. type IV).

3.2. PROPOSITION 3.1. *Let $G = < A_1, A_2 >$ be a Fuchsian Schottky group of type I, and let (t_1, t_2, ρ) be the point representing G. Let $d(G)$ be the Hausdorff dimension of the limit set of G. If $t_1 = t_2$ with $0 < t_1 < \sqrt{5} - 2$ and $\rho = -1/3$, then*

$$\frac{\log 3}{\frac{2r(1-r)}{1-2r} - \log \frac{r^2}{5+4\sqrt{1+r^2}+3r^2}} \leq d(G) \leq \frac{\log 3}{\log(1-r) - \log r},$$

where $r = 2\sqrt{t}/(1-t)$.

EXAMPLE. If $\rho = -1/3, t_1 = t_2 = 33 - 8\sqrt{17}$, then

$$0.2797 < d(E) \leq 0.5.$$

Bishop-Jones' theorem [2]. If $\{G_n\}$ is a sequence of N-generated Kleinian groups which converges algebraically to G, then

$$d(\Lambda(G)) \leq \liminf d(\Lambda(G_n)).$$

It suffices to consider the Hausdorff dimension of the limit sets of Schottky groups in a fundamental regions for the Schottky modular group acting on $R_I \mathfrak{S}_2^0$ and $R_{IV} \mathfrak{S}_2^0$ (see Sato [6]). By Proposition 3.1 and Bishop-Jones' theorem we have the following.

THEOREM 1.

(1) $\sup\{d(G) | G \in R_{IV} \mathfrak{S}_2^0\} = 1,$

(2) $\inf\{d(G) | G \in R_{IV} \mathfrak{S}_2^0\} = 0.$

THEOREM 2.

(1) $\sup\{d(G) | G \in R_I \mathfrak{S}_2^0\} \geq 1/2.$

(2) $\inf\{d(G) | G \in R_I \mathfrak{S}_2^0\} = 0.$

3.3. We end this paper by presenting some problems.

PROBLEM.

1. Given (t_1, t_2, ρ) corresponding to a classical Schottky group $G = < A_1, A_2 >$, represent the Hausdorff dimension of the limit set of G in terms of t_1, t_2 and ρ.

2. Find the best upper bound of the Hausdorff dimension of the limit set of classical Schottky groups of genus two (cf. Doyle [3]).

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